Luttinger liquid properties of highly correlated electron systems in one dimension

1991 J. Phys.: Condens. Matter 3 5983

(http://iopscience.iop.org/0953-8984/3/32/007)

J. Phys.: Condens. Matter 3 (1991) 5983–6008. Printed in the UK

Luttinger liquid properties of highly correlated electron systems in one dimension

Norio Kawakami and Sung-Kil Yang†
Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606, Japan

Received 20 August 1990, in final form 28 January 1991

Abstract. An exact description is given of the long-distance behaviour of the one-dimensional t–J model at $t = J$. We employ the Bethe *ansatz* method and the finite-size scaling technique in conformal field theory. The charge and spin degrees of freedom are separated, and described by two independent $c = 1$ conformal theories. The critical exponents for the charge, spin, electron and superconducting correlation functions are obtained for arbitrary band filling. We then make detailed comparison of the t–J model with the repulsive Hubbard model with emphasis on their Luttinger liquid properties. Analysing the electron filling dependence we observe the enhancement of the superconducting correlations compared with the highly correlated Hubbard model. The effect of the external magnetic field at and near half-filling is also discussed.

1. Introduction

Almost ten years ago Haldane introduced the concept of Luttinger liquids that is valid in understanding the low-energy behaviour of a large class of one-dimensional (1D) conducting fermion systems [1,2]. The universal role of Fermi liquids in higher dimensions is thus replaced by Luttinger liquids in one dimension. These two types of quantum fluids have quite distinct features. Among others, in ordinary Fermi liquid theory, well-defined propagation of electron quasiparticles implies a finite jump discontinuity in the momentum distribution function at the Fermi momentum. This should be contrasted with the power-law singularity near the Fermi point in Luttinger liquids, which corresponds to the soliton-like excitation instead of the quasiparticle excitation [3].

Recent studies of high-T_c superconductivity have renewed interest in low-dimensional electron systems. In superconducting compounds the quantum fluctuation inherent in low dimensions is believed to play a crucial role in addition to the strong correlation effect near the insulating phase [4]. To find an appropriate model of high-T_c superconductors it is of particular importance to clarify if non-Fermi liquid behaviour appears in the normal state of low-dimensional highly correlated systems. A fundamental model Hamiltonian to study such correlated systems may be provided by the Hubbard model, or more simplified t–J model. In one dimension these systems are the simplest examples which have been expected to possess a non-Fermi liquid nature. Very recently numerical computations to resolve this issue in the highly correlated Hubbard chain have been done by Sorella *et al* [5,6], Imada

† Present address: National Laboratory for High Energy Physics (KEK), Ibaraki 305, Japan.

0953-8984/91/325983+26$03.50

and Hatsugai [7], and Ogata and Shiba [8]. Their works motivated us to find exact correlation exponents in 1D correlated systems and to clarify their Luttinger liquid nature.

In this paper we will describe exactly the long-distance behaviour of various correlation functions in the 1D t–J model at $t = J$. In addition to a full exposition of the results announced in a previous communication [9] an analysis of the magnetic field effect is also reported. The results will be compared in detail with the properties of the 1D Hubbard model whose correlation exponents have also been obtained quite recently by Schulz [10], Kawakami and Yang [11], and Frahm and Korepin [12]. bosonization formulae for the Hubbard model are given by Affleck [13, 14]. These models have been known to be exactly solved by the Bethe ansatz for arbitrary electron filling [15, 16]. In the Bethe *ansatz* approach it is a formidable task to deal with correlation functions. However, recent developments in two-dimensional conformal field theory have made it possible to calculate the correlation exponents [17]. The point is that under a conformal mapping the scaling operators and the eigenstates of the transfer matrix on a finite periodic strip have a one-to-one correspondence [18]. Consequently the critical exponents are obtained if one knows the gap due to the finite-size effect in the spectrum of the Hamiltonian at criticality. On the other hand, computation of the energy gap is the most tractable problem in the Bethe ansatz, and hence we are able to compute exactly various correlation exponents based on the finite-size scaling analysis [19–23].

Interacting 1D quantum systems may carry several low-energy excitations with linear dispersion relations, but with different Fermi velocities. Hence the systems will not be Lorentz invariant. When the motions of these excitations are decoupled, however, we can still apply the conformal theory technique [22, 23]. This is indeed the case for the 1D t–J model and the Hubbard model, where the charge and spin degrees of freedom are separated in the continuum limit, as will be seen. Consequently the charge fluctuation is described by a $c = 1$ conformal theory with continuously varying exponents as functions of the electron filling. Here c is the central charge of the Virasoro algebra. The spin fluctuation belongs to the universality class of the antiferromagnetic spin-$\frac{1}{2}$ Heisenberg chain irrespective of the electron filling. This class is a well known $c = 1$ SU(2) Kac–Moody theory.

In section 2 we recapitulate the Bethe *ansatz* solutions to the 1D t–J model at $t = J$ and compute the finite-size corrections in the energy spectrum. The long-distance properties of the charge, spin, electron and superconducting correlation functions for arbitrary band filling are described in section 3. The magnetic field dependence of correlation exponents at and near half-filling is also studied. In section 4 we first review the properties of Luttinger liquids in the light of our result for the t–J model, and then make the comparison with the Hubbard model. The relationship between the critical exponents and the bulk quantities is also discussed. The final section is devoted to our conclusions. In appendices A and B we summarize some technical details.

2. Finite-size scaling behaviour of the energy spectrum

The 1D t–J model consists of spin-$\frac{1}{2}$ electrons hopping around nearest-neighbour lattice sites with the hopping matrix element $-t < 0$. We assume there is no double-occupancy of every site, reflecting a large on-site Coulomb repulsion. Furthermore the motion of highly correlated electrons (or holes) is supposed to be strongly affected by the spin fluctuation through the antiferromagnetic coupling $J > 0$. The Hamiltonian is then given by [4]

$$\mathcal{H} = -t\sum_{i,\sigma}(c_{i\sigma}^{\dagger}c_{i+1\sigma} + c_{i+1\sigma}^{\dagger}c_{i\sigma}) + 2J\sum_{i}(S_i \cdot S_{i+1} - \tfrac{1}{4}n_i n_{i+1})$$
$$-\mu\sum_i n_i - \frac{1}{2}H\sum_i(n_{i\uparrow} - n_{i\downarrow}) \tag{2.1}$$

where $c_{i\sigma}$ ($\sigma =\uparrow$ or $\downarrow$) is the spin-σ electron annihilation operator at the ith site, $S_i = c_{i\sigma}^{\dagger}S_{\sigma\sigma'}c_{i\sigma'}$ with the spin-$\frac{1}{2}$ matrix S, the number operator $n_{i\sigma} = c_{i\sigma}^{\dagger}c_{i\sigma}$, $n_i = n_{i\uparrow} + n_{i\downarrow}$, and μ and H are the chemical potential and the external magnetic field, respectively.

As is well known, this Hamiltonian is formally obtained by the canonical transformation from the Hubbard model, but with the limitation $J \ll t$. In (2.1), however, one can regard t and J as free parameters. Therefore the model we shall treat here ($t = J$) has an extremely large exchange coupling compared with the strong correlation limit of the Hubbard model. The relevance of such a model to the high-T_c superconductivity was first demonstrated by Zhang and Rice [24]. Subsequently many attempts have been made to clarify the nature of the t–J model, in particular laying stress upon the competition between magnetism and superconductivity.

Schlottmann found that the 1D t–J model (2.1) can be solved by the Bethe ansatz for the special case of $t = J$ [15]. At this integrable point the model is mapped onto the multicomponent quantum lattice gas whose exact solution was obtained by Sutherland [25]. The diagonalization is performed in two steps. First we seek for the wave function as a superposition of the plane waves characterized by the electron momenta p_j ($j = 1 \sim N_c$). Here we consider a 1D lattice of even N sites with N_c electrons among which M electrons are spin down. The complete integrability is then ensured by the factorization of the multiparticle scattering matrix (Yang–Baxter relation). On applying periodic boundary conditions we reduce the problem to the ancillary one in spin space. This problem can be solved by the generalized Bethe ansatz by introducing the spin rapidity Λ_α ($\alpha = 1 \sim M$) related to the internal degrees of freedom. The resulting Bethe–Yang transcendental equations are written in terms of the rapidities $k_j = \frac{1}{2}\cot(p_j/2)$ and Λ_α [25, 15]

$$\left(\frac{k_j + \mathrm{i}/2}{k_j - \mathrm{i}/2}\right)^N = \prod_{\beta=1}^{M}\frac{k_j - \Lambda_\beta + \mathrm{i}/2}{k_j - \Lambda_\beta - \mathrm{i}/2} \qquad j = 1,\ldots,N_c$$
$$\prod_{j=1}^{N_c}\frac{\Lambda_\alpha - k_j + \mathrm{i}/2}{\Lambda_\alpha - k_j - \mathrm{i}/2} = -\prod_{\beta=1}^{M}\frac{\Lambda_\alpha - \Lambda_\beta + \mathrm{i}}{\Lambda_\alpha - \Lambda_\beta - \mathrm{i}} \qquad \alpha = 1,\ldots,M. \tag{2.2}$$

For convenience we will set $t = J = 1$ from this point on.

The set of rapidities $\{k_j\}$ contains complex $k_\alpha^{\pm}$ of spin paired electrons ($\alpha = 1 \sim M$), where $k_\alpha^{\pm}$ are determined by real (down-) spin rapidities Λ_α through $k_\alpha^{\pm} = \Lambda_\alpha \pm \mathrm{i}/2$ [15]. At first sight the complex solutions $k_\alpha^{\pm}$ seem to generate the charge excitation gap as in the attractive Hubbard model. It turns out, however, that they describe the massless charge excitation except for the half-filled band, where

$N_c = N$ [26]. We note that similar observation was first made in the singlet ground state of the Anderson model for the Kondo problem [27]. The real solutions k_j describe the spin excitation at zero temperature. Other string solutions for excited states are not necessary for the present investigation.

Substituting the complex solution into (2.2) and taking the logarithm we obtain [15]

$$2N\tan^{-1}(2k_j) = 2\pi I_j + 2\sum_{\beta=1}^{M}\tan^{-1}(2(k_j - \Lambda_\beta)) \qquad j = 1,\ldots,N_c - 2M \tag{2.3}$$

$$\begin{aligned}2N\tan^{-1}(\Lambda_\alpha) = 2\pi J_\alpha &+ 2\sum_{j=1}^{N_c-2M}\tan^{-1}(2(\Lambda_\alpha - k_j)) \\ &+2\sum_{\beta=1}^{M}\tan^{-1}(\Lambda_\alpha - \Lambda_\beta) \qquad \alpha = 1,\ldots,M\end{aligned} \tag{2.4}$$

where

$$I_j = \frac{M}{2} \bmod 1 \qquad J_\alpha = \frac{N_c + M + 1}{2} \bmod 1\,. \tag{2.5}$$

The energy and the momentum are given by

$$\begin{aligned}E &= -2\sum_{j=1}^{N_c}\cos p_j - \mu N_c + H\left(M - \frac{N_c}{2}\right) \\ &= -2N_c + 2\sum_{j=1}^{N_c-2M}\frac{1/2}{k_j^2 + 1/4} \\ &\quad + 2\sum_{\alpha=1}^{M}\frac{1}{\Lambda_\alpha^2 + 1} - \mu N_c + H\left(M - \frac{N_c}{2}\right)\end{aligned} \tag{2.6}$$

$$P = \sum_{j=1}^{N_c} p_j = -\frac{2\pi}{N}\left(\sum_{j=1}^{N_c-2M} I_j + \sum_{\alpha=1}^{M} J_\alpha\right). \tag{2.7}$$

It is convenient to introduce

$$z_{s,N}(k) = \frac{1}{2\pi}\left(2\tan^{-1}(2k) - \frac{1}{N}\sum_{\beta=1}^{M}2\tan^{-1}(2(k - \Lambda_\beta))\right) \tag{2.8}$$

$$\begin{aligned}z_{c,N}(\Lambda) = \frac{1}{2\pi}\bigg(&2\tan^{-1}(\Lambda) - \frac{1}{N}\sum_{j=1}^{N_c-2M}2\tan^{-1}(2(\Lambda - k_j)) \\ &- \frac{1}{N}\sum_{\beta=1}^{M}2\tan^{-1}(\Lambda - \Lambda_\beta)\bigg)\end{aligned} \tag{2.9}$$

$$\rho_{s,N}(k) = \frac{\partial z_{s,N}(k)}{\partial k} \qquad \rho_{c,N}(\Lambda) = \frac{\partial z_{c,N}(\Lambda)}{\partial \Lambda} \tag{2.10}$$

so that

$$z_{s,N}(k_j) = \frac{I_j}{N} \qquad z_{c,N}(\Lambda_\alpha) = \frac{J_\alpha}{N}. \tag{2.11}$$

Inspecting (2.11) one finds that the real solutions $\{k_j\}$ and $\{\Lambda_\alpha\}$ distribute over the regions $k < B^-$ and $k > B^+$, $\Lambda < Q^-$ and $\Lambda > Q^+$, respectively. Correspondingly the distributions of the quantum numbers I_j and J_α become $I < I^-$ and $I > I^+$, $J < J^-$ and $J > J^+$, where

$$z_{s,N}(B^\pm) = \frac{I^\pm}{N} \qquad z_{c,N}(Q^\pm) = \frac{J^\pm}{N} \tag{2.12}$$

and

$$\begin{aligned} I^+ - I^- &= N - N_s \qquad & I^+ + I^- &= 2D_s \\ J^+ - J^- &= N - N_c \qquad & J^+ + J^- &= 2D_c. \end{aligned} \tag{2.13}$$

Here $N_s = N_c - M$ is the number of up spins and D_s (or D_c) denotes the number of particles which transfer from a Fermi level of the spinon (or holon) to the other Fermi level.

2.1. *Corrections to the ground-state energy*

We now take the large-N limit while keeping the terms which scale as $1/N$ in the energy spectrum [19]. First from (2.10) we get

$$\rho_{s,N}(k) = \frac{1}{2\pi}\left(T_{cs}(k) - \frac{1}{N}\sum_{\beta=1}^{M} T_{sc}(k - \Lambda_\beta)\right) \tag{2.14}$$

$$\rho_{c,N}(\Lambda) = \frac{1}{2\pi}\left(T_{cc}(\Lambda) - \frac{1}{N}\sum_{j=1}^{N_c-2M} T_{cs}(\Lambda - k_j) - \frac{1}{N}\sum_{\beta=1}^{M} T_{cc}(\Lambda - \Lambda_\beta)\right) \tag{2.15}$$

where

$$T_{sc}(x) = T_{cs}(x) = \frac{1}{x^2 + 1/4} \qquad T_{cc}(x) = \frac{2}{x^2+1} \qquad T_{ss}(x) \equiv 0. \tag{2.16}$$

Using the Euler–Maclaurin formula

$$\frac{1}{N}\sum_{n=n_1}^{n_2} f\left(\frac{n}{N}\right) = \int_{(n_1-1/2)/N}^{(n_2+1/2)/N} f(x)\,\mathrm{d}x - \frac{1}{24N^2}\left(f'(\frac{n_2+1/2}{N}) - f'(\frac{n_1-1/2}{N})\right) \tag{2.17}$$

we obtain from (2.14) and (2.15)

$$\begin{aligned} \rho_{\alpha,N}(\lambda_\alpha) = {} & \frac{1}{2\pi}\tilde{a}_\alpha(\lambda_\alpha) + \frac{1}{24N^2}\sum_\beta \left(\frac{T'_{\alpha\beta}(\lambda_\alpha - q^+_\beta)}{2\pi\rho_{\beta,N}(q^+_\beta)} - \frac{T'_{\alpha\beta}(\lambda_\alpha - q^-_\beta)}{2\pi\rho_{\beta,N}(q^-_\beta)}\right) \\ & - \sum_{\beta=c,s}\int_{\pm\beta}\frac{\mathrm{d}\lambda'}{2\pi}T_{\alpha\beta}(\lambda_\alpha - \lambda')\rho_{\beta,N}(\lambda') \qquad \alpha = c, s \end{aligned} \tag{2.18}$$

where we have introduced the notation $\lambda_c = \Lambda$, $\lambda_s = k$, $q_c^{\pm} = Q^{\pm}$, $q_s^{\pm} = B^{\pm}$, $\tilde{a}_s(x) = T_{cs}(x)$, $\tilde{a}_c(x) = T_{cc}(x)$ and the integral

$$\int_{\pm\beta} = \int_{q_\beta^+}^{+\infty} + \int_{-\infty}^{q_\beta^-} .$$

The solution to (2.18) may be written as

$$\rho_{\alpha,N}(\lambda_\alpha) = \rho_\alpha(\lambda_\alpha|q^{\pm}) + \frac{1}{24N^2}\sum_\beta \left(\frac{f_{\alpha\beta}^+(\lambda_\alpha|q^{\pm})}{2\pi\rho_{\beta,N}(q_\beta^+)} - \frac{f_{\alpha\beta}^-(\lambda_\alpha|q^{\pm})}{2\pi\rho_{\beta,N}(q_\beta^-)}\right) \tag{2.19}$$

where

$$\rho_\alpha(\lambda_\alpha|q^{\pm}) = \frac{1}{2\pi}\tilde{a}_\alpha(\lambda_\alpha) - \sum_\beta \int_{\pm\beta} \frac{\mathrm{d}\lambda'}{2\pi} T_{\alpha\beta}(\lambda_\alpha - \lambda')\rho_\beta(\lambda'|q^{\pm}) \tag{2.20}$$

$$f_{\alpha\beta}^{\pm}(\lambda_\alpha|q^{\pm}) = T'_{\alpha\beta}(\lambda_\alpha - q_\beta^{\pm}) - \sum_\gamma \int_{\pm\gamma} \frac{\mathrm{d}\lambda'}{2\pi} T_{\alpha\gamma}(\lambda_\alpha - \lambda') f_{\gamma\beta}^{\pm}(\lambda'|q^{\pm}) . \tag{2.21}$$

Notice that $\rho_\alpha(\lambda_\alpha|q^{\pm})$ in (2.20) are the rapidity distribution functions in the thermodynamic limit $N \to \infty$ with $N_c/N = n_c$ and $N_s/N = n_s$ being kept fixed. In this limit the electron density n_c and the magnetization $\mathcal{M}$ are obviously given by

$$n_c = \int_{\pm s} \mathrm{d}k\, \rho_s(k) + 2\int_{\pm c} \mathrm{d}\Lambda\, \rho_c(\Lambda)$$
$$\mathcal{M} = n_s - \frac{n_c}{2} = \frac{1}{2}\int_{\pm s} \mathrm{d}k\, \rho_s(k) . \tag{2.22}$$

For the ground state the rapidity distribution is symmetric, $q_\alpha^{\pm} = \pm q_\alpha$. In the absence of the external magnetic field the ground state turns out to be singlet, $\mathcal{M} = 0$ [15].

To calculate the energy we first apply the formula (2.17) to (2.6). Then, using (2.19) and (2.21), we have

$$E = N\epsilon(q^{\pm}) - \frac{\pi}{6N}\sum_\alpha \frac{1}{2\pi\rho_\alpha(q_\alpha)}\left(- \tilde{\epsilon}_\alpha^{0\prime}(q_\alpha) - \sum_\beta \int_{|\lambda|\geqslant q_\beta} \frac{\mathrm{d}\lambda}{2\pi} \tilde{\epsilon}_\beta^0(\lambda) f_{\beta\alpha}^+(\lambda)\right)$$
$$+ \mathrm{O}(N^{-2}) \tag{2.23}$$

where

$$\epsilon(q^{\pm}) = \sum_\alpha \int_{\pm\alpha} \mathrm{d}\lambda\, \tilde{\epsilon}_\alpha^0(\lambda)\rho_\alpha(\lambda|q^{\pm}) \tag{2.24}$$

$$\tilde{\epsilon}_s^0(\lambda) = -(2+\mu) - \frac{H}{2} + T_{cs}(\lambda) \qquad \tilde{\epsilon}_c^0(\lambda) = -2(2+\mu) + T_{cc}(\lambda) . \tag{2.25}$$

Since the second term in (2.23) is of order N^{-1} we have replaced $\rho_\alpha(\lambda|q^{\pm})$ by $\rho_\alpha(\lambda) = \lim_{q_\beta^{\pm}\to\pm q_\beta} \rho_\alpha(\lambda|q^{\pm})$ which is the solution to (2.20) with $q_\beta^{\pm}$ being replaced by $\pm q_\beta$. Likewise $f_{\alpha\beta}^+(\lambda_\alpha) = \lim_{q_\gamma^{\pm}\to\pm q_\gamma} f_{\alpha\beta}^+(\lambda_\alpha|q^{\pm})$.

Let us introduce the dressed energy functions

$$\epsilon_\alpha(\lambda|q^\pm) = -\tilde{\varepsilon}^0_\alpha(\lambda) - \sum_\beta \int_{\pm\beta} \frac{\mathrm{d}\lambda'}{2\pi} \epsilon_\beta(\lambda'|q^\pm) T_{\beta\alpha}(\lambda' - \lambda) \tag{2.26}$$

with the condition

$$\epsilon_\alpha(q^\pm_\alpha|q^\pm) = 0\,. \tag{2.27}$$

We iterate (2.26) and take the derivative. The result is compared with the expression obtained by inserting the iteration solution $f^+_{\alpha\beta}$ of (2.21) into (2.23). One finds

$$\epsilon'_\alpha(q_\alpha) \equiv \frac{\partial}{\partial\lambda}\epsilon_\alpha(\lambda)\bigg|_{\lambda=q_\alpha} = -\tilde{\varepsilon}^{0\prime}_\alpha(q_\alpha) - \sum_\beta \int_{|\lambda|\geqslant q_\beta} \frac{\mathrm{d}\lambda}{2\pi} \tilde{\varepsilon}^0_\beta(\lambda) f^+_{\beta\alpha}(\lambda) \tag{2.28}$$

where $\epsilon_\alpha(\lambda) = \lim_{q^\pm_\beta \to \pm q_\beta} \epsilon_\alpha(\lambda|q^\pm)$. On the other hand, the Fermi velocities of the low-lying excitations are determined from

$$v_\alpha = \frac{1}{2\pi\rho_\alpha(q_\alpha)}\epsilon'_\alpha(q_\alpha)\,. \tag{2.29}$$

Therefore it can be seen that

$$E = N\varepsilon_\alpha(q^\pm) - \frac{\pi v_s}{6N} - \frac{\pi v_c}{6N} + \mathrm{O}(N^{-2})\,. \tag{2.30}$$

Consequently we find that the ground-state energy scales as

$$E_0 = N\varepsilon_0 - \frac{\pi v_s}{6N} - \frac{\pi v_c}{6N} + \mathrm{O}(N^{-2}) \tag{2.31}$$

where the bulk energy density $\varepsilon_0 = \varepsilon(q^\pm)|_{q^\pm_\alpha = \pm q_\alpha} \equiv \varepsilon(\pm q)$.

2.2. Corrections due to the excitations

Our next task is to compute the energy gap $E - E_0$ due to the elementary excitations. There exist two types of excitations: the excitations which cause the change of the symmetric Fermi level $\pm B$ and $\pm Q$ of the ground state to the asymmetric ones $B^\pm$ and $Q^\pm$, thereby with large momentum transfer, and the particle–hole excitations with small momentum transfer near the Fermi levels.

In order to calculate the contribution of the excitations with large momentum transfer, it is convenient to convert the integrals $\int_{\pm\gamma}$ into $\int_{q^-_\gamma}^{q^+_\gamma}$. This can be performed by Fourier transform. The integral equation (2.20) for the rapidity distribution then turns out to be

$$\rho_\alpha(\lambda|q^\pm) = \frac{1}{2\pi}a_\alpha(\lambda) + \sum_\beta \int_{q^-_\beta}^{q^+_\beta} \frac{\mathrm{d}\lambda'}{2\pi} K_{\alpha\beta}(\lambda - \lambda')\rho_\beta(\lambda'|q^\pm) \tag{2.32}$$

where $K_{cs}(x) = K_{sc}(x) = T_{sc}(x)$, $K_{ss}(x) = -T_{cc}(x)$, $K_{cc}(x) \equiv 0$, and $a_s(x) = K_{sc}(x)$, $a_c(x) = 0$. For the dressed energy (2.26) we obtain

$$\epsilon_\alpha(\lambda|q^\pm) = \epsilon^0_\alpha(\lambda) + \sum_\beta \int_{q_\beta^-}^{q_\beta^+} \frac{\mathrm{d}\lambda'}{2\pi} \epsilon_\beta(\lambda'|q^\pm) K_{\beta\alpha}(\lambda' - \lambda) \tag{2.33}$$

where $\epsilon^0_s(x) = H - K_{cs}(x)$, $\epsilon^0_c(x) = 2 + \mu - H/2$. The energy takes the form

$$\varepsilon(q^\pm) = \mu - \frac{H}{2} + \sum_\alpha \int_{q_\alpha^-}^{q_\alpha^+} \mathrm{d}\lambda\, \epsilon^0_\alpha(\lambda) \rho_\alpha(\lambda|q^\pm)\,. \tag{2.34}$$

The integrations of the rapidity distributions over the closed intervals yield

$$\int_{Q^-}^{Q^+} \mathrm{d}\Lambda\, \rho_c(\Lambda) = 1 - n_c \qquad \int_{B^-}^{B^+} \mathrm{d}k\, \rho_s(k) = 1 - n_s \tag{2.35}$$

which are the number of holes and the number of 'holes' with respect to up spins, respectively. Notice that these are quite consistent with (2.13).

We turn now to the derivation of the explicit form of $\varepsilon(q^\pm)$. We first minimize $\varepsilon(q^\pm)$ with respect to the electron number and the magnetization. This condition is equivalent to demanding $\partial\varepsilon(q^\pm)/\partial q^\pm_\alpha = 0$, which is realized by virtue of the condition (2.27) for the dressed energy; $\epsilon_\alpha(q^\pm_\alpha|q^\pm) = 0$. See appendix A. Let us next expand $\varepsilon(q^\pm)$ about the ground-state energy density $\varepsilon_0 = \varepsilon(\pm q)$

$$\varepsilon(q^\pm) = \varepsilon_0 + \frac{1}{2}\sum_\alpha \left\{ \left(\frac{\partial}{\partial q^+_\alpha}\right)^2 \varepsilon \Big| (q^+_\alpha - q_\alpha)^2 + \left(\frac{\partial}{\partial q^-_\alpha}\right)^2 \varepsilon \Big| (q^-_\alpha + q_\alpha)^2 \right\} \tag{2.36}$$

where the vertical bar stands for setting $q^\pm_\alpha = \pm q_\alpha$. There is no cross derivative due to (2.27). We now wish to express the variations $dq^\pm_\alpha = q^\pm_\alpha \mp q_\alpha$ in terms of the change of the numbers of electrons and up spins. The details of this calculation are left to appendix A. The result reads

$$\varepsilon(q^\pm) - \varepsilon_0 = \frac{2\pi}{N^2}\left(\frac{1}{4} I^t (\mathbf{Z}^{-1})^t \mathbf{V} \mathbf{Z}^{-1} I + D^t \mathbf{Z} \mathbf{V} \mathbf{Z}^t D\right) + \mathrm{O}(N^{-2}) \tag{2.37}$$

where $\mathbf{V} = \mathrm{diag}(v_c, v_s)$. Here we have introduced the 2×2 dressed charge matrix $\mathbf{Z}$ [22,23,28] whose elements $Z_{\alpha\beta} = \xi_{\alpha\beta}(q_\beta)$ are given by the solutions to the integral equations

$$\xi_{\alpha\beta}(\lambda_\beta) = \delta_{\alpha\beta} + \sum_{\gamma=c,s} \int_{-q_\gamma}^{q_\gamma} \frac{\mathrm{d}\lambda}{2\pi} \xi_{\alpha\gamma}(\lambda) K_{\gamma\beta}(\lambda - \lambda_\beta)\,. \tag{2.38}$$

Here

$$D = \begin{pmatrix} D_c \\ D_s \end{pmatrix} \qquad I = \begin{pmatrix} I_c \\ I_s \end{pmatrix} = \begin{pmatrix} -(N_c - n^0_c N) \\ -(N_s - n^0_s N) \end{pmatrix} \tag{2.39}$$

with n^0_α being the ground-state value of $n_\alpha, \alpha = c, s$. As pointed out in [28,29] n^0_c, n^0_s and N should meet certain commensuration conditions to be consistent with the conformal limit. I_c and I_s are then non-negative integers, implying that we are counting the hole number and the number of 'holes' with respect to the up spins, respectively.

It is straightforward to include the particle–hole excitations. Their contributions are specified by the set of non-negative integers $N_c^{\pm}$ and $N_s^{\pm}$. The final expression for the energy gap is thus obtained as

$$E - E_0 = \frac{2\pi v_c}{N} x_c + \frac{2\pi v_s}{N} x_s + \mathrm{O}(N^{-2}) \tag{2.40}$$

$$\begin{aligned} x_c &= \left(\frac{Z_{ss}I_c - Z_{cs}I_s}{2\det \mathbf{Z}}\right)^2 + (Z_{cc}D_c + Z_{sc}D_s)^2 + N_c^+ + N_c^- \\ x_s &= \left(\frac{Z_{sc}I_c - Z_{cc}I_s}{2\det \mathbf{Z}}\right)^2 + (Z_{cs}D_c + Z_{ss}D_s)^2 + N_s^+ + N_s^- . \end{aligned} \tag{2.41}$$

The momentum takes the form

$$P - P_0 = (2\pi - 2k_{F\uparrow} - 2k_{F\downarrow})D_c + (2\pi - 2k_{F\uparrow})D_s + \frac{2\pi}{N} \sum_{\alpha=c,s} (I_\alpha D_\alpha + N_\alpha^+ - N_\alpha^-) \tag{2.42}$$

where P_0 is the ground-state momentum and the Fermi momentum $k_{F\uparrow}$ ($k_{F\downarrow}$) for the up- (down)-spin electrons is given by

$$k_{F\uparrow(\downarrow)} = \tfrac{1}{2}\pi(n_c \pm 2\mathcal{M}) . \tag{2.43}$$

Equation (2.42) is easily checked if one notes to rewrite (2.7) as

$$P = \frac{2\pi}{N}\left(\sum I_j + \sum J_\alpha\right) + (I^{\pm}, J^{\pm} \text{ independent term}) \tag{2.44}$$

where the sums are taken over $I_j \in [I^-, I^+]$ and $J_\alpha \in [J^-, J^+]$.

This completes our derivation of the finite-size corrections in the energy spectrum. Now, conformal invariance of 1D quantum critical systems dictates that the ground-state energy scales like [30]

$$E_0 = \varepsilon_0 N - \frac{\pi c}{6N} v + \mathrm{O}(N^{-1}) \tag{2.45}$$

where v is the Fermi velocity and c is the central charge of the Virasoro algebra. The energy gaps of the excited states are related to the scaling dimensions x_n of the scaling operators of the theory [18]

$$E_n - E_0 = \frac{2\pi v}{N} x_n + \mathrm{O}(N^{-1}) . \tag{2.46}$$

Thus our expressions (2.31) and (2.40) indicate that the critical behaviour of the t–J model is described by two independent $c = 1$ conformal theories. They are associated with the massless excitations, the holon and spinon, which are characterized by the Fermi velocities v_c and v_s.

3. Correlation functions

One of the remarkable observations in two-dimensional conformal field theory is that the critical exponents of the scaling operators are read off from the energy gaps as described in (2.46). To write down explicitly the correlation functions at long distance let us rewrite (2.40) and (2.42) as

$$E(I,D) - E_0 = \frac{2\pi}{N} \sum_{\alpha=c,s} v_\alpha(\Delta_\alpha^+ + \Delta_\alpha^-) + O(N^{-1}) \tag{3.1}$$

$$P(I,D) - P_0 = (2\pi - 2k_{F\uparrow} - 2k_{F\downarrow})D_c + (2\pi - 2k_{F\uparrow})D_s + \frac{2\pi}{N} \sum_{\alpha=c,s} (\Delta_\alpha^+ - \Delta_\alpha^-) \tag{3.2}$$

where $\Delta_\alpha^\pm$ are the left and right conformal weights in the sector α; $\alpha = c$ (holon), $\alpha = s$ (spinon). Here $x_\alpha = \Delta_\alpha^+ + \Delta_\alpha^-$ and we have

$$\Delta_c^\pm(I,D) = \frac{1}{2}\left(Z_{cc}D_c + Z_{sc}D_s \pm \frac{Z_{ss}I_c - Z_{cs}I_s}{2\det \mathbf{Z}} \right)^2 + N_c^\pm \tag{3.3}$$

$$\Delta_s^\pm(I,D) = \frac{1}{2}\left(Z_{cs}D_c + Z_{ss}D_s \pm \frac{Z_{cc}I_s - Z_{sc}I_c}{2\det \mathbf{Z}} \right)^2 + N_s^\pm . \tag{3.4}$$

The two-point correlation functions of the scaling fields $\phi_{\Delta^\pm}(x,t)$ with conformal weights $\Delta^\pm$ then take the form

$$\begin{aligned} \langle \phi_{\Delta^\pm}(x,t)\phi_{\Delta^\pm}(0,0)\rangle &\equiv G(\Delta^\pm(I,D)|x,t) \\ &= \frac{\exp(i(2\pi - 2k_{F\uparrow} - 2k_{F\downarrow})D_c x)\exp(i(2\pi - 2k_{F\uparrow})D_s x)}{(x - iv_c t)^{2\Delta_c^+}(x + iv_c t)^{2\Delta_c^-}(x - iv_s t)^{2\Delta_s^+}(x + iv_s t)^{2\Delta_s^-}} . \end{aligned} \tag{3.5}$$

We consider the following correlation functions.

(i) Electron correlator

$$G_\sigma(x,t) = \langle c_\sigma^\dagger(x,t)c_\sigma(0,0)\rangle \qquad \sigma = \uparrow \text{ or } \downarrow . \tag{3.6}$$

(ii) Charge density correlator

$$N(x,t) = \langle n(x,t)n(0,0)\rangle \qquad n(x,t) = n_\uparrow(x,t) + n_\downarrow(x,t) . \tag{3.7}$$

(iii) Spin correlator

$$\chi(x,t) = \langle S_z(x,t)S_z(0,0)\rangle \qquad S_z(x,t) = \tfrac{1}{2}(n_\uparrow(x,t) - n_\downarrow(x,t)) . \tag{3.8}$$

(iv) Singlet and triplet pair superconducting correlators

$$\begin{aligned} P_s(x,t) &= \langle c_\uparrow^\dagger(x+1,t)c_\downarrow^\dagger(x,t)c_\uparrow(1,0)c_\downarrow(0,0)\rangle \\ P_t(x,t) &= \langle c_\uparrow^\dagger(x+1,t)c_\uparrow^\dagger(x,t)c_\uparrow(1,0)c_\uparrow(0,0)\rangle . \end{aligned} \tag{3.9}$$

Generically the field operators we have introduced will renormalize to a certain linear combination of the scaling operators at long distance. The correlation functions (3.6)–(3.9) are thus expressed as

$$\sum A(I, D, N^{\pm}) G(\Delta^{\pm}(I, D)|x, t) \tag{3.10}$$

where $A(I, D, N^{\pm})$ are constant coefficients and we have neglected possible logarithmic corrections.

In order to determine the scaling dimensions we now have to assign the quantum numbers $(I, D, N^{\pm})$ to the field operators, as has been done for the Hubbard mode [12]. Notice that these quantum numbers are subject to the restrictions

$$D_c = \frac{I_c + I_s}{2} \bmod 1 \qquad D_s = \frac{I_c}{2} \bmod 1 \tag{3.11}$$

which can be checked from (2.5). Upon inspecting the explicit form of the field operators in (3.6)–(3.9) one finds the assignment

$$\begin{aligned}
&G_\uparrow(x,t) : (I_c = 1, I_s = 1, D_c \in \mathbb{Z}, D_s \in \mathbb{Z} + \tfrac{1}{2}) \\
&G_\downarrow(x,t) : (I_c = 1, I_s = 0, D_c \in \mathbb{Z} + \tfrac{1}{2}, D_s \in \mathbb{Z} + \tfrac{1}{2}) \\
&N(x,t) : (I_c = 0, I_s = 0, D_c \in \mathbb{Z}, D_s \in \mathbb{Z}) \\
&\chi(x,t) : (I_c = 0, I_s = 0, D_c \in \mathbb{Z}, D_s \in \mathbb{Z}) \\
&P_s(x,t) : (I_c = 2, I_s = 1, D_c \in \mathbb{Z} + \tfrac{1}{2}, D_s \in \mathbb{Z}) \\
&P_t(x,t) : (I_c = 2, I_s = 2, D_c \in \mathbb{Z}, D_s \in \mathbb{Z}) .
\end{aligned} \tag{3.12}$$

In the following we first study the correlation functions for zero magnetic field and then the effect of the external magnetic field at and near half-filling is discussed.

3.1. Zero magnetic field

It is readily seen that $B \to +\infty$ for zero magnetic field. Using the Fourier transform technique we obtain the simple form of the dressed charge matrix $\mathbf{Z}$

$$\begin{pmatrix} Z_{cc} & Z_{cs} \\ Z_{sc} & Z_{ss} \end{pmatrix} = \begin{pmatrix} \xi_c(Q) & 0 \\ \xi_c(Q)/2 & 1/\sqrt{2} \end{pmatrix} \tag{3.13}$$

where $Z_{ss} = 1/\sqrt{2}$ is derived with the aid of the Wiener–Hopf method [28]. Here $\xi_c(\Lambda)$ is the solution to the equation

$$\xi_c(\Lambda) = 1 + \int_{-Q}^{Q} \mathrm{d}\Lambda' \, R(\Lambda - \Lambda') \xi_c(\Lambda') \tag{3.14}$$

with the kernel being

$$R(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \frac{\exp(-\mathrm{i}\omega x)}{1 + \exp|\omega|} . \tag{3.15}$$

The conformal weights (3.3) and (3.4) are reduced to

$$\begin{aligned} \Delta_c^{\pm}(\boldsymbol{I},\boldsymbol{D}) &= \frac{1}{2}\left(\frac{I_c}{2\xi_c(Q)} \pm \xi_c(Q)(D_c + \frac{D_s}{2})\right)^2 + N_c^{\pm} \\ \Delta_s^{\pm}(\boldsymbol{I},\boldsymbol{D}) &= \frac{1}{4}\left(I_s - \frac{I_c}{2} \mp D_s\right)^2 + N_s^{\pm}\,. \end{aligned} \tag{3.16}$$

Let us first consider the charge density correlation function. From (3.12) we write down the asymptotic form of the equal-time correlator

$$N(r,0) \sim \text{constant} + A_0 r^{-2} + A_2 r^{-\alpha_s}\cos(2k_F r) + A_4 r^{-\alpha_c}\cos(4k_F r) \tag{3.17}$$

where $k_{F\uparrow} = k_{F\downarrow} \equiv k_F$ since $\mathcal{M} = 0$ for zero field. The $4k_F$ piece arises from the excitation of $(I_c, I_s, D_c, D_s) = (0,0,\pm1,0)$, while the $2k_F$ piece from $(I_c, I_s, D_c, D_s) = (0,0,\pm1,\mp1)$ and $(0,0,0,\pm1)$. The non-oscillating part is due to the lowest particle–hole excitation. We thus find

$$\alpha_c = 2\xi_c(Q)^2 \qquad \alpha_s = 1 + \alpha_c/4\,. \tag{3.18}$$

Notice that both the holon and spinon excitations are responsible for the $2k_F$ oscillation part. On the other hand the $4k_F$ piece is dominated by the holon excitation alone. The same observation holds for the Hubbard model [10–12] and the Tomonaga–Luttinger model [3]. The spin correlation function $\chi(r,0)$ has the same form as (3.17) except that the $4k_F$ part is absent. The critical exponent for the $2k_F$ part is equal to α_s of the charge density correlation.

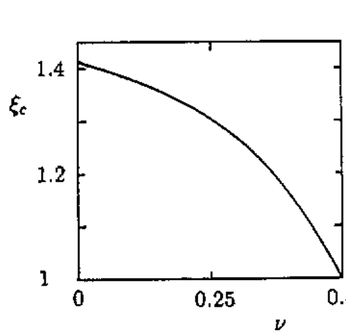

Figure 1. The dressed charge $\xi_c(Q)$ of the holon as a function of the electron concentration ν ($\nu = \frac{1}{2}$ for half-filling).

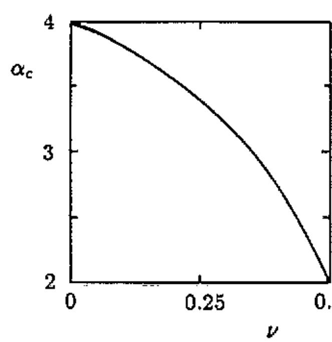

Figure 2. The charge density $4k_F$ exponent α_c as a function of ν.

The dressed charge $\xi_c(Q)$ of the holon is shown in figure 1, where $\nu = n_c^0/2$ and $\nu = \frac{1}{2}$ corresponds to the half-filled band. The $4k_F$ exponent α_c then behaves as depicted in figure 2. Near half-filling we obtain $\alpha_c \sim 2 + 8(\frac{1}{2} - \nu)$ as shown in appendix B. Note that in the low-density limit $\alpha_c = 4$, i.e. the value for the non-interacting model.

The long-distance behaviour of the electron correlation function is governed by the excitation specified by $(I_c, I_s, D_c, D_s) = (1,1,0,\pm\frac{1}{2})$. We thus obtain

$$G_\uparrow(r,0) \sim r^{-\eta}\cos(k_F r) \qquad \eta = (\alpha_c + 4)^2/(16\alpha_c)\,. \tag{3.19}$$

$G_{\downarrow}(r,0)$ follows the same behaviour, but with the excitation $(1,0,\pm\frac{1}{2},\mp\frac{1}{2})$. Consequently the momentum distribution function close to k_F has the form

$$\langle n_k\rangle = \langle n_{k_F}\rangle - \mathrm{constant}\,|k-k_F|^\theta \mathrm{sgn}(k-k_F) \tag{3.20}$$

which is the typical power-law singularity of the Luttinger liquid [3] and we find

$$\theta = \eta - 1 = (\alpha_c - 4)^2/(16\alpha_c)\,. \tag{3.21}$$

From figure 3 we see that as ν deviates from half-filling θ decreases monotonically from $\frac{1}{8}$ to zero, and hence the momentum distribution in the low-density regime exhibits an abrupt change around k_F.

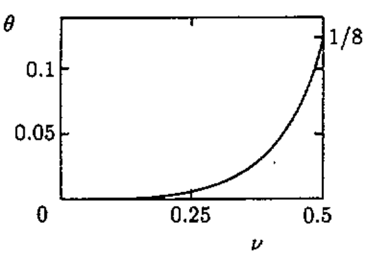

Figure 3. The exponent θ for the momentum distribution as a function of ν.

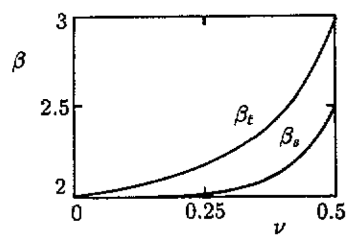

Figure 4. The superconducting correlation exponents as a function of ν. β_s and β_t are for the singlet and triplet pair, respectively.

We now turn to the superconducting correlation functions. The excitations relevant to the singlet and triplet pair correlations are specified by $(I_c, I_s, D_c, D_s) = (2,1,\pm\frac{1}{2},0)$ and $(2,2,0,0)$, respectively. We then obtain for the singlet pair

$$P_s(r,0) \sim r^{-\beta_s}\cos(2k_F r) \qquad \beta_s = 4/\alpha_c + \alpha_c/4\,. \tag{3.22}$$

The triplet pair has the leading uniform term

$$P_t(r,0) \sim r^{-\beta_t} \qquad \beta_t = 1 + 4/\alpha_c\,. \tag{3.23}$$

Notice that the singlet pair correlation also has the uniform piece with the same exponent β_t. The exponents β_s and β_t are plotted in figure 4, from which we observe that the superconducting correlations get more enhanced as holes are doped into the half-filled band [9, 31]. It is interesting to notice that even in the t–J model the superconducting correlations never overwhelm the spin correlation since β_t and β_s are always larger than α_s for arbitrary electron filling.

3.2. *Magnetic field dependence*

Let us investigate how the correlation exponents behave when we turn on the external magnetic field. For simplicity we consider two typical cases: just at half-filling and near half-filling, on the basis of which we will be able to clarify the essential properties of the field dependence. At half-filling there is no massless excitation associated with the charge fluctuation since the strong correlation effect opens the very large Hubbard

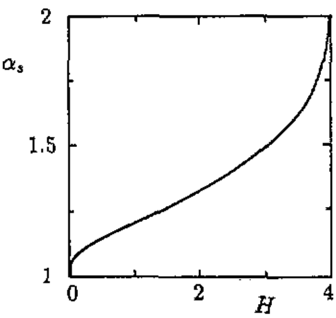

Figure 5. The $2k_F$ exponent α_s in the spin correlator as a function of H at half-filling.

gap. The spin excitation remains massless, which can be described by the $c = 1$ SU(2) Kac–Moody theory. The long-distance behaviour of the spin correlator is thus equivalent to that for the antiferromagnetic Heisenberg model in [32]

$$\chi(r,0) \sim \mathcal{M}^2 + B_0 r^{-2} + B_2 r^{-\alpha_s} \cos(2k_{F\downarrow} r) \tag{3.24}$$

where $\mathcal{M}$ is the magnetization. We plot in figure 5 the magnetic field dependence of the $2k_F$ spin exponent $\alpha_s = 2\xi_s(B)^2$, where $\xi_s(B)$ is given in (3.26) below.

In the metallic phase away from half-filling, the holon becomes massless as in zero field. An essential difference from zero-field case is that the holon is no longer treated as a spinless hole because it acquires the effective spin induced by the magnetic field. Similarly the spinon may get electrically charged.

These effective spin and charge are computed by creating the holon and spinon excitations in magnetic fields [33]. We then observe that they are nothing but the elements of the dressed charge matrix introduced in section 2. The physical meaning of each element is that Z_{cc} and $(\frac{1}{2} - Z_{sc})$ are the effective charge and spin of the holon, and Z_{ss} and Z_{cs} are the effective spin and charge of the spinon. Approaching half-filling ($Q \to 0$), the effective charge of the holon is independent of field and becomes unity. The spinon is not charged even in the metallic phase, i.e. $Z_{cs} = 0$. Furthermore the field dependence of the effective spin of the spinon is given by that for the Heisenberg model (corresponding to half-filling). The dressed charge matrix thus turns out to be

$$\begin{pmatrix} Z_{cc} & Z_{cs} \\ Z_{sc} & Z_{ss} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} - s_h(B) & \xi_s(B) \end{pmatrix} \tag{3.25}$$

where $\xi_s(B)$ is equal to the dressed charge (or effective spin) for the spin-$\frac{1}{2}$ Heisenberg chain obeying

$$\xi_s(k) = \frac{1}{2} + \int_{|k'|\geqslant B} \mathrm{d}k'\, R(k-k')\xi_s(k') \tag{3.26}$$

with the kernel $R(x)$ given in (3.15) and the effective spin of the holon is

$$s_h(B) = \frac{1}{2}\int_{|k|\geqslant B} \mathrm{d}k\, \mathrm{sech}\,(\pi k)\xi_s(k)\,. \tag{3.27}$$

For comparison let us quote the zero-field dressed charge matrix near half-filing

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1/\sqrt{2} \end{pmatrix} . \tag{3.28}$$

It is not difficult to verify (3.25) from (2.38) if one applies Fourier transform. The field dependence of $\xi_s(B)$ and $s_h(B)$ are depicted in figures 6 and 7, respectively. We note in figures 5 and 6 that the weak-field behaviour of α_s and ξ_s exhibit the logarithmic singularity whose origin is the same as for the spin susceptibility of the Heisenberg chain [12,34].

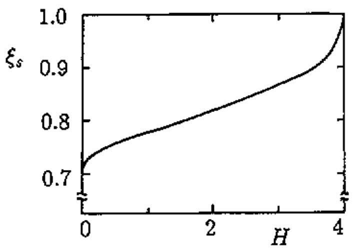

Figure 6. The dressed charge $\xi_s(B)$ of the spinon as a function of the external magnetic field H at and near half-filling.

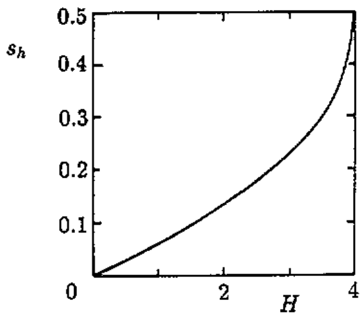

Figure 7. The effective spin $s_h(B)$ of the holon as a function of H near half-filling.

We next discuss the field dependence of critical exponents close to half-filling. The exponent of the $4k_F(= 2k_{F\uparrow} + 2k_{F\downarrow})$ oscillation piece in the charge correlator takes the value $\alpha_c = 2$ irrespective of magnetic fields owing to the fact that it is controlled by charge excitation alone. Since the charge density operator $n(r) = n_\uparrow(r) + n_\downarrow(r)$ the $2k_F$ part splits into two pieces with the momentum $2k_{F\downarrow}$ and $2k_{F\uparrow}$, the exponents of which are given by $\alpha_{s\downarrow} = 2Z_{ss}^2 + 2(1 - Z_{sc})^2$ and $\alpha_{s\uparrow} = 2Z_{ss}^2 + 2Z_{sc}^2$, respectively. Note that these exponents have the magnetic-field dependence only through the effective spins of the spinon and holon. The values of $(\alpha_{s\downarrow}, \alpha_{s\uparrow})$ are increased from $(\frac{3}{2}, \frac{3}{2})$ to $(4, 2)$ as the field increases.

The singlet pairing exponent has the form

$$\beta_s = \frac{5}{2} + \frac{1}{2}\left(\frac{1 - 2Z_{sc}}{Z_{ss}}\right)^2 \tag{3.29}$$

while the triplet one reads

$$\beta_t = 2 + 2\left(\frac{1 - Z_{sc}}{Z_{ss}}\right)^2 . \tag{3.30}$$

With the increase of the magnetic field, the values of (β_s, β_t) are monotonically increased from $(\frac{5}{2}, 3)$ to $(3, 4)$, respectively. Therefore the superconducting correlation is suppressed in the presence of the magnetic field, as might be expected.

The Luttinger anomaly exponent for the momentum distribution is given by

$$\begin{aligned} \theta_{\uparrow} &= \frac{1}{2}\left[-1+Z_{sc}^2+Z_{ss}^2+\left(\frac{1-Z_{sc}}{Z_{ss}}\right)^2\right] && \text{for } k_{F\uparrow} \\ \theta_{\downarrow} &= \frac{1}{2}\left[-1+(1-Z_{sc})^2+Z_{ss}^2+\left(\frac{Z_{sc}}{Z_{ss}}\right)^2\right] && \text{for } k_{F\downarrow}\,. \end{aligned} \tag{3.31}$$

Both of the exponents $\theta_{\uparrow,\downarrow}$ increase up to $\frac{1}{2}$ with the increase of magnetic fields. The momentum distribution around the Fermi momentum is therefore smoothed by the magnetic field.

We mention that Ogata *et al* analysed numerical data on the magnetic field dependence in the $U \to \infty$ limit of the Hubbard model [35] comparing with the analytic result obtained by Frahm and Korepin [12]. Our present results for the magnetic field effect in the t–J model are essentially the same as theirs in the metallic system very close to half-filling. It is worth noting that in this regime the effective spin of the holon defined here is nothing but the magnetization $\mathcal{M}$ (2.22) of the system. The magnetization in this limit is of course equal to that in the Heisenberg model calculated by Griffiths [34]. We point out, however, that this relation holds only for highly correlated systems. In generic cases they are not equivalent. This will be seen explicitly in subsection 4.1, where the effect of the finite Coulomb interaction is discussed using the Hubbard model.

4. Luttinger liquid properties

According to Haldane, the idea of Luttinger liquids applies to the low-energy excitations in a variety of 1D metallic systems [1,2]. His demonstration is mainly based on the systems containing the one-component massless excitation, such as the Heisenberg model, the bose gas model *etc.* The low-energy spectrum of Luttinger liquids contains the three spectral parameters, $v_{\rm F}$, v_J and v_N. These are all velocities associated with the excitations of particle–hole pairs ($v_{\rm F}$), of the $2k_{\rm F}$ momentum transfer (v_J), and of the particle number change (v_N). Here $v_{\rm F}$ is the usual sound velocity. The crucial point is that these velocities are not mutually independent but connected through the universal relation $v_{\rm F} = (v_J v_N)^{1/2}$. Hence one can write

$$v_J = \exp(2\psi)v_{\rm F} \qquad v_N = \exp(-2\psi)v_{\rm F} \tag{4.1}$$

where the parameter $\exp(2\psi)$ is non-universal and depends on the details of the interactions of underlying microscopic models. All the correlation exponents are essentially determined by this parameter. In short, what Haldane claims is that the low-energy massless excitations in 1D metallic systems are all solved by the procedure of bosonization.

In the t–J model, as we have seen, there exist two massless degrees of freedom, the holon and spinon. These excitations are decoupled and described by two independent $c = 1$ conformal theories, i.e. Gaussian theories. Thus the low-energy action reads

$$S = \frac{1}{2}\sum_{\alpha=c,s} v_\alpha \int {\rm d}t \int_0^{2\pi} {\rm d}x\, \partial_\mu\phi_\alpha\partial_\mu\phi_\alpha \tag{4.2}$$

where free boson fields are periodic $\phi_\alpha(t,x) = \phi_\alpha(t,x+2\pi)+2\pi N_\alpha R_\alpha$ with $N_\alpha \in \mathbf{Z}/2$. The conformal weights (3.16) are characteristic of the Gaussian theory [36]. For the charge sector, therefore, the field periodicity R_c is parametrized as $\sqrt{\pi}R_c = \xi_c(Q)^{-1}$, i.e. it depends continuously on the electron concentration. The spin sector has the periodicity $\sqrt{\pi}R_s = Z_{ss} = 1/\sqrt{2}$ for any electron density. This implies that the spin sector is described by the level-1 SU(2) Kac–Moody theory just like the spin-$\frac{1}{2}$ antiferromagnetic Heisenberg chain [13]. It is also instructive to compare the formula (3.1) for the energy gaps $2\pi v_\alpha(\Delta_\alpha^+ + \Delta_\alpha^-)/N$ with Haldane's result (see equation (6) of [2]). They are in fact equivalent under the identification $e^\psi = \xi_c(Q)$ (or $e^\psi = Z_{ss} = 1/\sqrt{2}$) for the charge (or spin) sector. In the presence of the magnetic field the quantity e^ψ is generalized to the dressed charge matrix. Hence the critical properties of the t–J model nicely fit in with the Luttinger liquid picture.

In comparison with the Fermi liquid theory the most striking feature of the Luttinger liquid is the power-law singularity of the momentum distribution function (3.20) near $k = k_F$. This reflects the fact that the low-energy excitation is not of the quasiparticle type, but of the collective type. The power-law anomaly (in view of the Fermi liquid theory) was first discovered in the Tomonaga–Luttinger model which essentially describes a weakly correlated electron system [37–39]. As for highly correlated systems this behaviour has been established only recently in the repulsive Hubbard model [6, 8, 10–12, 40]. We now have shown that the same conclusion holds for the t–J model.

In order for these systems to be classified as Luttinger liquids it has been crucial that the charge and spin degrees of freedom are separated and described by two independent $c = 1$ conformal field theories. Universal scaling relations (3.18), (3.21) and (3.23) are then valid for these metallic models. Each exponent, however, depends on the non-universal microscopic property of the theory due to the existence of the marginal operator. To clarify this point we would like to compare the t–J model with the repulsive Hubbard model in the next subsection.

4.1. Comparison with the Hubbard model

The 1D Hubbard chain describes a system of itinerant electrons feeling the on-site Coulomb repulsion U. The Hamiltonian takes the form

$$\mathcal{H} = -t\sum_{i,\sigma}(c_{i\sigma}^\dagger c_{i+1\sigma} + c_{i+1\sigma}^\dagger c_{i\sigma}) + U\sum_i n_{i\uparrow}n_{i\downarrow} \qquad U > 0\,. \tag{4.3}$$

As mentioned before, in the strong correlation limit ($U \gg t$) the model reduces to the t–J model in the region $J \sim 2t^2/U \ll t$.

The finite-size corrections in the Hubbard model have been analysed by Woynarovich [28]. For vanishing magnetic field the critical exponents α_s, θ, β_s and β_t are all expressed in terms of α_c just through the same scaling relations (3.18) and (3.21)–(3.23) as in the t–J model [10–12]. The $4k_F$ oscillation exponent α_c is determined through $\alpha_c = 2\eta_c(Q)^2$, where the dressed charge function $\eta_c(k)$ of the holon is the solution to the integral equation [11, 12]

$$\eta_c(k) = 1 + \int_{-Q}^{Q} \mathrm{d}k' \cos(k') G(\sin k - \sin k')\eta_c(k') \tag{4.4}$$

with the kernel being

$$G(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \frac{\exp(-i\omega x)}{1+\exp(U|\omega|/(2t))}\,. \tag{4.5}$$

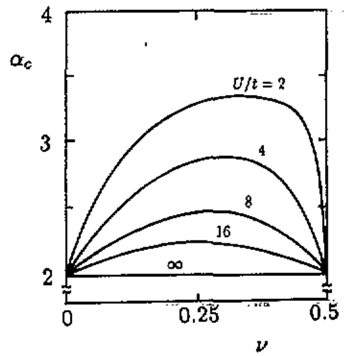

Figure 8. The charge density $4k_F$ exponent α_c as a function of ν in the Hubbard model.

Here the Fermi level Q is fixed by the electron concentration.

In figure 8 we show the exponent α_c. Strong dependence of α_c on the Coulomb interaction as well as the electron filling is clearly observed. As $U \to \infty$, α_c approaches 2 in agreement with the result for the spinless fermion. In the opposite limit $U \to 0$, α_c converges to 4 for the electron concentration $0 < \nu < \frac{1}{2}$, which is consistent with the result of the Tomonaga–Luttinger model. It should be noticed that α_c takes the value close to 2 near half-filling as long as the Coulomb interaction exists. Recall that at half-filling the Hubbard model is an insulator for all $U \neq 0$, since the Umklapp interaction becomes relevant, thereby the charge excitation possesses the gap. The gap formation strongly affects the properties of the charge excitation so that the holon behaves like the spinless fermion, resulting in the $\alpha_c = 2$ near half-filling. From α_c one can evaluate the Luttinger anomaly exponent θ for the momentum distribution and the superconducting correlation exponents through (3.21)–(3.23) [10–12]. The results are plotted in figures 9 and 10.

Let us compare the present result for the t–J model with the large-U behaviour of the Hubbard model. In the vicinity of the half-filled band the exponents of the t–J model take the values expected in the strong correlation limit of the Hubbard model, for instance $\alpha_c = 2$. This is because the exclusion of the double occupation gives the most dominant effect near half-filling, which makes the motion of doped holes behave like spinless fermions as in the Hubbard model. In the $U \to \infty$ Hubbard model, as ν decreases from half-filling α_c $(= 2)$ stays constant, and hence $\theta = \frac{1}{8}$ for any filling [10–12, 41]. On the other hand, in the t–J model the critical exponents take the values for the non-interacting system such as α_c $(= 4)$ in the low-density limit $\nu \to 0$.

This non-interacting behaviour of the t–J model for $\nu \to 0$ seems to be a bit peculiar since the model is originally supposed to describe a highly correlated system. Our result implies that the hole motion in the t–J model is not like spinless fermions for large hole-doping, but is considerably influenced by the spin fluctuation through the strong antiferromagnetic coupling J. We think that the large antiferromagnetic coupling favours the antiparallel-spin electron pairs to sit on the nearest-neighbour sites, which renders the hole motion quite different from spinless particles. In the low-density limit this configuration will be so dominant that the exclusion of the double occupancy becomes less important.

Turning to the superconducting correlations we see that the large spin coupling

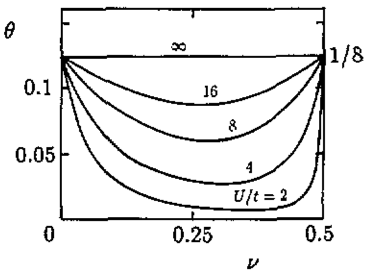

Figure 9. The exponent θ for the momentum distribution as a function of ν in the Hubbard model.

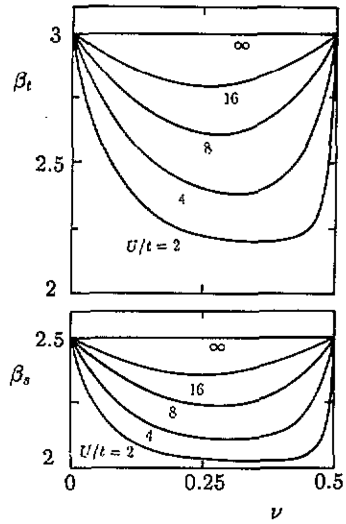

Figure 10. The superconducting correlation exponents as a function of ν in the Hubbard model. β_s and β_t are for the singlet and triplet pair, respectively.

as well as the hole doping in the t–J model play a conspicuous role to enhance the superconducting correlation. This is not the case for the strong correlation limit of the Hubbard model. Thus the t–J model tends to stabilize the superconducting state. In spite of this fact, however, the spin correlation always dominates the superconducting correlations for arbitrary electron filling, as pointed out in section 3.

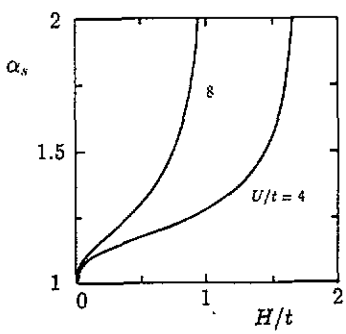

Figure 11. The $2k_F$ exponent α_s in the spin correlator as a function of H in the Hubbard model at half-filling.

Finally we discuss the magnetic field dependence. In [11, 12] the exponent α_s for the $2k_F$ oscillation piece in the spin correlator just at half-filling has been expressed as $\alpha_s = 2\eta_s(B)^2$, where $\eta_s(B)$ is the dressed charge explained below. We present the field dependence of α_s in figure 11. In the vicinity of the half-filled band, the dressed charge matrices for zero and for non-zero field take the same form as (3.25) and (3.28). The dressed charge function (or effective spin) η_s of the spinon satisfies the integral equation (3.26) but with the kernel replaced by $G(x)$ defined in (4.5). The effective spin $\eta_s(B)$ of the spinon is plotted in figure 12 for several values of U/t. We also depict the effective spin s_h of the holon in figure 13.

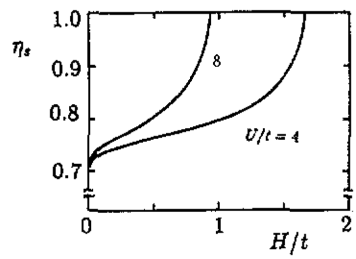

Figure 12. The dressed charge (or effective spin) $\eta_s(B)$ of the spinon as a function of H in the Hubbard model at and near half-filling.

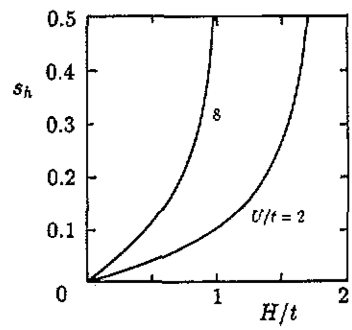

Figure 13. The effective spin $s_h(B)$ of the holon as a function of H in the Hubbard model near half-filling (In this figure $U/t = 2$ should read $U/t = 4$).

All the critical exponents are obtained in terms of the effective spins of the spinon and holon. We shall refrain from giving explicit formulae since one can readily check the field dependence of exponents using the formulae given in subsection 3.2.

Let us conclude this section by making a brief comment on the effective spin s_h of the holon. For $U/t \gg 1$ the s_h curve closely follows the magnetization curve of the Heisenberg chain, as observed in the t–J model. This observation is understood in the following way: In the strongly correlated regime near half-filling the spin state is almost degenerate, and hence all the band electrons contribute equally to the magnetization under non-zero field. Therefore making a hole in the ground-state Λ-distribution amounts to losing magnetization per lattice site. This in turn gives rise to the effective spin of the holon. Notice, however, that such a simple situation no longer holds as U/t becomes small. Therefore it should be realized that the effective spin of the holon has a different field dependence from the magnetization generically.

4.2. Relations to bulk quantities

Another interesting aspect of the Luttinger liquids is that the critical exponents can be expressed in terms of the bulk quantities. This kind of relation between the bulk quantity and the dressed charge was first noticed in [1,32]. We consider the three typical bulk quantities, the spin susceptibility χ_s, the compressibility χ_c and the specific heat coefficient γ, in the t–J model.

As shown in appendix B, the compressibility and the spin susceptibility are obtained as

$$\chi_c = \xi_c(Q)^2/(\pi v_c)$$
$$\chi_s = (g\mu_B)^2\xi_s^2/(\pi v_s) \qquad \xi_s = 1/\sqrt{2}. \tag{4.6}$$

The low-temperature expansion of the free energy gives

$$\gamma = \frac{\pi}{3}\left(\frac{1}{v_c} + \frac{1}{v_s}\right) \tag{4.7}$$

which corresponds to two $c = 1$ conformal theories [30]. We thus find

$$\alpha_c = 4\tilde{\chi}_c/(2\tilde{\gamma} - \tilde{\chi}_s) \tag{4.8}$$

where we have renormalized the bulk quantities so that $\tilde{\gamma} = \tilde{\chi}_s = \tilde{\chi}_c = 1$ in the non-interacting limit. Note that this formula is also valid for the Hubbard model [10–12]. In the band bottom all these bulk quantities exhibit the divergent behaviour due to the dispersion relation in 1D electron systems. Approaching half-filling χ_s remains finite (a constant value of the Heisenberg model), while χ_c diverges as

$$\chi_c \simeq \frac{8(\ln 2)^2}{3\pi^2\zeta(3)}(\tfrac{1}{2} - \nu)^{-1} \tag{4.9}$$

due to the diverging density of states (see appendix B) [42], where ζ is the Riemann zeta function. Since γ is also divergent like $(\frac{1}{2} - \nu)^{-1}$ we have $\alpha_c \to 2\tilde{\chi}_c/\tilde{\gamma}$ for $\nu \to \frac{1}{2}$.

Let us next discuss an important role played by boundary conditions. Imposing twisted boundary conditions on the Bethe wavefunction does not ruin the exact integrability by virtue of the $U(1)$ symmetry of the system. Shastry and Sutherland then noticed that this was an efficient way to evaluate the effective current-carrying mass (transport mass) [43]. Under twisted boundary conditions with the twisting phase ϕ the shift of the ground-state energy from the periodic case (ϕ=0) is

$$E_0(\phi) - E_0(0) = \mathcal{D}_c\phi^2/N + O(\phi^4)\,. \tag{4.10}$$

The interesting point is that the charge stiffness $\mathcal{D}_c$ is directly related to the DC part of the conductivity $\sigma(\omega)$

$$\mathrm{Re}\,\sigma(\omega) = \frac{2\pi e^2}{\hbar}\mathcal{D}_c\delta(\hbar\omega)\,. \tag{4.11}$$

For free electrons the coefficient of $\delta(\hbar\omega)$ is proportional to m^{-1} with m being the electron mass. Therefore it is legitimate to define the effective mass m^* through $m^*/m \propto \mathcal{D}_c^{-1}$ [43]. In view of conformal theories the energy shift due to twisted boundary conditions by ϕ is attributed to the excitation $I_c = I_s = D_s = 0$ and $D_c = \phi/2\pi$ [12]. From (3.1), (3.16) and (4.6)–(4.8) one can easily express the enhancement factor of the current-carrying mass in terms of the bulk quantities [44]

$$m^*/m = (2\tilde{\gamma} - \tilde{\chi}_s)^2/\tilde{\chi}_c\,. \tag{4.12}$$

Then, for instance, in the t–J model near half-filling the effective mass is extremely enhanced as

$$m^*/m \sim \frac{8(\ln 2)^2}{3\zeta(3)}(\tfrac{1}{2} - \nu)^{-1} \tag{4.13}$$

which corresponds to the fact that the system approaches the insulating phase.

To conclude this section we emphasize that formulae (4.8) and (4.12) are valid for any 1D correlated electron system, and hence characterize the universal properties of Luttinger liquids.

5. Conclusions

In this paper the long-distance properties of the t–J model at $t = J$ for arbitrary electron filling have been studied using the Bethe *ansatz* solution and the finite-size scaling method in conformal theory. The results are compared with the repulsive Hubbard model in detail. Starting with microscopic models we have shown explicitly that the electron behaviour in these highly correlated systems is characterized as the Luttinger liquid. The separation of the charge and spin degrees of freedom is quite essential. Consequently the charge sector is described by the Gaussian theory and the spin sector by the $c = 1$ SU(2) current algebra. The dressed charge matrix introduced in the Bethe *ansatz* calculation provides us with the precise link between the characteristic parameter of Luttinger liquids (i.e. Gaussian field periodicity) and the microscopic parameters in the theory. Notice that this is the most difficult step in a conventional bosonization approach. In conclusion we have presented the microscopic foundation of the concept of Luttinger liquids á la Haldane on the basis of conformal field theory and Bethe *ansatz* solutions.

Acknowledgments

We thank H Frahm and V E Korepin for useful communications.

Appendix A

In this appendix we present our calculation of the finite-size corrections in section 2 in such a way that it can be applied to generic nested Bethe *ansatz* solutions. Let us start with the Bethe *ansatz* equations

$$Np^0_\alpha(\lambda_j) = 2\pi I^\alpha_j - \sum_{\beta=1}^{l}\sum_{k=1}^{N_\beta}\phi_{\alpha\beta}(\lambda^\alpha_j - \lambda^\beta_k) \qquad \alpha = 1,\ldots,l \qquad j = 1,\ldots,N_\alpha \tag{A1}$$

where N denotes the system size, N_α is the number of 'particles' of the type α $(= 1,\ldots,l)$ and $p^0_\alpha(\lambda)$ are the bare momenta. The phase shifts $\phi_{\alpha\beta}(\lambda)$ are assumed to obey $\phi_{\alpha\beta}(\lambda) = \phi_{\beta\alpha}(\lambda) = -\phi_{\alpha\beta}(-\lambda)$. We consider the case in which the quantum numbers I^α_j belong to the interval $[I^+_\alpha, I^-_\alpha]$ so that

$$I^+_\alpha - I^-_\alpha = N_\alpha \qquad I^+_\alpha + I^-_\alpha = 2D_\alpha\,. \tag{A2}$$

Define

$$z_{\alpha,N}(\lambda) = \frac{1}{2\pi}p^0_\alpha(\lambda) + \frac{1}{2\pi N}\sum_{\beta=1}^{l}\sum_{k=1}^{N_\beta}\phi_{\alpha\beta}(\lambda - \lambda^\beta_k) \tag{A3}$$

$$\rho_{\alpha,N}(\lambda) = \frac{\partial z_{\alpha,N}(\lambda)}{\partial\lambda} \qquad z_{\alpha,N}(q^\pm_\alpha) = \frac{I^\pm_\alpha}{N}\,. \tag{A4}$$

For $N \to \infty$ with $N_\alpha / N = \nu_\alpha$ and $D_\alpha / N = \delta_\alpha$ being fixed, the rapidity distribution functions satisfy

$$\rho_\alpha(\lambda | q^\pm) = \frac{1}{2\pi} a_\alpha(\lambda) + \sum_\beta \int_{q_\beta^-}^{q_\beta^+} \frac{\mathrm{d}\lambda'}{2\pi} K_{\alpha\beta}(\lambda - \lambda') \rho_\beta(\lambda' | q^\pm) \tag{A5}$$

where $a_\alpha(\lambda) = p_\alpha^{0\prime}(\lambda)$ and $K_{\alpha\beta}(\lambda) = \phi'_{\alpha\beta}(\lambda)$. Then, for $z_\alpha(\lambda) = \lim_{N\to\infty} z_{\alpha,N}(\lambda)$, we have

$$z_\alpha(\lambda) = \frac{1}{2\pi} p_\alpha^0(\lambda) + \sum_\beta \int_{q_\beta^-}^{q_\beta^+} \frac{\mathrm{d}\lambda'}{2\pi} \phi_{\alpha\beta}(\lambda - \lambda') \rho_\beta(\lambda' | q^\pm). \tag{A6}$$

Let us first calculate

$$\frac{\partial \nu_\alpha}{\partial q_\beta^+} = \frac{\partial}{\partial q_\beta^+} (z_\alpha(q_\alpha^+) - z_\alpha(q_\alpha^-)) = \frac{\partial}{\partial q_\beta^+} \int_{q_\alpha^-}^{q_\alpha^+} \mathrm{d}\lambda \, \rho_\alpha(\lambda | q^\pm) . \tag{A7}$$

Introducing the dressed charge functions

$$\xi_{\alpha\beta}(\lambda_\beta) = \delta_{\alpha\beta} + \sum_\gamma \int_{-q_\gamma}^{q_\gamma} \frac{\mathrm{d}\lambda'}{2\pi} \xi_{\alpha\gamma}(\lambda') K_{\gamma\beta}(\lambda' - \lambda_\beta) \tag{A8}$$

we obtain

$$\left. \frac{\partial \nu_\alpha}{\partial q_\beta^+} \right| = \rho_\beta(q_\beta) Z_{\alpha\beta} \tag{A9}$$

where the vertical bar is meant to put $q_\alpha^\pm = \pm q_\alpha$ and the $l \times l$ dressed charge matrix $\mathbf{Z}$ is given by $Z_{\alpha\beta} = \xi_{\alpha\beta}(q_\beta)$.

We next calculate

$$2 \frac{\partial \delta_\alpha}{\partial q_\beta^+} = \frac{\partial z_\alpha(q_\alpha^+)}{\partial q_\beta^+} + \frac{\partial z_\alpha(q_\alpha^-)}{\partial q_\beta^-} . \tag{A10}$$

After some manipulations we get

$$\left. \frac{\partial \delta_\alpha}{\partial q_\beta^+} \right| = \rho_\beta(q_\beta) \tfrac{1}{2} (\mathbf{Z}^t)^{-1}_{\alpha\beta} . \tag{A11}$$

In a similar way one can show that

$$\left. \frac{\partial \nu_\alpha}{\partial q_\beta^-} \right| = - \left. \frac{\partial \nu_\alpha}{\partial q_\beta^+} \right| \qquad \left. \frac{\partial \delta_\alpha}{\partial q_\beta^-} \right| = \left. \frac{\partial \delta_\alpha}{\partial q_\beta^+} \right| . \tag{A12}$$

We thus find in matrix notation that

$$\begin{pmatrix} \rho \mathrm{d}q^+ \\ \rho \mathrm{d}q^- \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\mathbf{Z}^{-1} & \mathbf{Z}^t \\ -\frac{1}{2}\mathbf{Z}^{-1} & \mathbf{Z}^t \end{pmatrix} \begin{pmatrix} \mathrm{d}\nu \\ \mathrm{d}\delta \end{pmatrix} . \tag{A13}$$

where $(\rho \mathrm{d}q^{\pm})_\alpha = \rho_\alpha(q_\alpha)\mathrm{d}q_\alpha^{\pm}(\mathrm{d}\nu)_\alpha = \mathrm{d}\nu_\alpha$ and $(\mathrm{d}\delta)_\alpha = \mathrm{d}\delta_\alpha$.

We are now ready to express the finite-size corrections in the energy

$$\varepsilon = \sum_\alpha \int_{q_\alpha^-}^{q_\alpha^+} \mathrm{d}\lambda\, \epsilon_\alpha^0(\lambda)\rho_\alpha(\lambda|q^{\pm}) \tag{A14}$$

in terms of the matrix **Z**. Let us define the dressed energy functions

$$\epsilon_\alpha(\lambda|q^{\pm}) = \epsilon_\alpha^0(\lambda) + \sum_\beta \int_{q_\beta^-}^{q_\beta^+} \frac{\mathrm{d}\lambda'}{2\pi}\epsilon_\beta(\lambda'|q^{\pm})K_{\beta\alpha}(\lambda' - \lambda) \tag{A15}$$

with the condition

$$\epsilon_\alpha(q_\alpha^{\pm}|q^{\pm}) = 0\,. \tag{A16}$$

This condition ensures the stationary condition

$$0 = \frac{\partial\varepsilon}{\partial q_\alpha^{\pm}} = \pm\epsilon_\alpha(q_\alpha^{\pm}|q^{\pm})\rho_\alpha(q_\alpha^{\pm}|q^{\pm})\,. \tag{A17}$$

Another basic relations are

$$\frac{1}{\rho_\alpha(q_\alpha)^2}\left(\frac{\partial}{\partial q_\alpha^{\pm}}\right)^2 \varepsilon \bigg| = 2\pi v_\alpha \tag{A18}$$

where v_α are the Fermi velocities.

Expanding $\varepsilon(q^{\pm})$ to second order in $\mathrm{d}q_\alpha^{\pm} = q_\alpha^{\pm} \mp q_\alpha$ and substituting (A13) and (A18) we finally obtain

$$\varepsilon(q^{\pm}) = \varepsilon(\pm q) + 2\pi\left(\frac{1}{4}(\mathrm{d}\nu)^t(\mathbf{Z}^{-1})^t\mathbf{V}\mathbf{Z}^{-1}\mathrm{d}\nu + (\mathrm{d}\delta)^t\mathbf{Z}\mathbf{V}\mathbf{Z}^t\mathrm{d}\delta\right) \tag{A19}$$

where $V_{\alpha\beta} = v_\alpha\delta_{\alpha\beta}$. Note that $\mathrm{d}\delta_\alpha = D_\alpha/N$ and $\mathrm{d}\nu_\alpha = N_\alpha/N - \nu_\alpha^0$ where ν_α^0 is the value for the ground state. Hence $\epsilon(q^{\pm}) - \epsilon(\pm q)$ is of order N^{-2}.

Appendix B

We express the compressibility $\chi_c = \partial n_c^0/\partial\mu$ in terms of the dressed charge $\xi_c(Q)$ in (3.14). First notice the chain rule

$$\frac{\partial n_c^0}{\partial\mu} = \frac{\partial n_c^0}{\partial Q}\frac{\partial Q}{\partial\mu}\,. \tag{B1}$$

For zero magnetic field ($B \to +\infty$) the rapidity distribution in the ground state satisfies

$$\rho_c(\Lambda) = R(\Lambda) + \int_{-Q}^{Q} \mathrm{d}\Lambda' R(\Lambda - \Lambda')\rho_c(\Lambda')\,. \tag{B2}$$

With the aid of an auxiliary function $F(\Lambda|\Lambda')$ obeying

$$F(\Lambda|\Lambda') = R(\Lambda - \Lambda') + \int_{-Q}^{Q} \mathrm{d}\nu R(\Lambda - \nu) F(\nu|\Lambda') \tag{B3}$$

we find from (3.14) and (B2) that

$$\begin{aligned} \xi_c(\Lambda) &= 1 + \int_{-Q}^{Q} F(\Lambda|\Lambda')\,\mathrm{d}\Lambda' \\ \frac{\partial \rho_c(\Lambda)}{\partial Q} &= \rho_c(Q)(F(\Lambda|Q) + F(\Lambda|-Q)) . \end{aligned} \tag{B4}$$

It is now straightforward to show from (2.35) that

$$\frac{\partial n_c^0}{\partial Q} = -2\rho_c(Q)\xi_c(Q) . \tag{B5}$$

The dressed energy function (2.33) for zero field obeys

$$\epsilon_c(\Lambda) = 2 + \mu - 2\pi R(\Lambda) + \int_{-Q}^{Q} \mathrm{d}\Lambda' R(\Lambda - \Lambda')\epsilon_c(\Lambda') . \tag{B6}$$

This function is subject to the condition $\epsilon_c(\pm Q) = 0$, according to which we obtain $\xi_c(Q) = -(\partial Q/\partial\mu)\epsilon_c'(Q)$. Using (B5) and (2.29) we thus verify the relation for χ_c in (4.6). The expression for the spin susceptibility $\chi_s = \partial\mathcal{M}/\partial H$ in (4.6) can be derived in a similar way by examining the asymptotic behaviour for $B \gg 1$.

Let us now check (4.9). Approaching half-filling we have $Q \to 0$, and hence from (2.35)

$$n_c^0 \simeq 1 - 2Q\rho_c(0) . \tag{B7}$$

Equation (B2) yields $\rho_c(0) \simeq R(0)$. Thus $Q \simeq (1 - n_c^0)/(2R(0))$. The dressed charge behaves as $\xi_c(Q) \simeq 1 + (1 - n_c^0)$. Similarly it is seen from (B6) that $\epsilon_c'(Q) \simeq -\pi(1 - n_c^0)R''(0)/R(0)$. After all this we get

$$\chi_c \simeq -\frac{2}{\pi}\frac{R(0)^2}{R''(0)}(1 - n_c^0)^{-1}. \tag{B8}$$

Inserting $R(0) = (1/\pi)\ln 2$, $R''(0) = -(3/2\pi)\zeta(3)$ with ζ being the Riemann zeta function, and $\nu = n_c^0/2$ we obtain (4.9) in the text.

References

[1] Haldane F D M 1980 *Phys. Rev. Lett.* **45** 1358; 1981 *J. Phys. C: Solid State Phys.* **14** 2585; 1981 *Phys. Lett.* **81A** 153

[2] Haldane F D M 1981 *Phys. Rev. Lett.* **47** 1840

[3] See, for reviews:
Solyom J 1979 *Adv. Phys.* **28** 201

Emery V J 1979 *Highly Conducting One-Dimensional Solids* ed J T Devreese *et al* (New York: Plenum) p 247
Fukuyama H and Takayama H 1985 *Electronic Properties of Inorganic Quasi-One-Dimensional Compounds* ed P Monceau (Dordrecht: Reidel) p 41
[4] Anderson P W 1987 *Science* **235** 1196
[5] Sorella S, Tosatti E, Baroni S, Car R and Parrinello M 1988 *Int. J. Mod. Phys.* B **2** 993
[6] Sorella S, Parola A, Parrinello M and Tosatti E 1990 *Europhys. Lett.* **12** 721
[7] Imada M and Hatsugai Y 1989 *J. Phys. Soc. Japan* **58** 3752
[8] Ogata M and Shiba H 1990 *Phys. Rev.* B **41** 2326
[9] Kawakami N and Yang S-K 1990 *Phys. Rev. Lett.* **65** 2309
[10] Schulz H J 1990 *Phys. Rev. Lett.* **64** 2831
[11] Kawakami N and Yang S-K 1990 *Phys. Lett.* **148A** 359
[12] Frahm H and Korepin V E 1990 *Phys. Rev.* B **42** 10553
[13] Affleck I 1989 *Fields, Strings and Critical Phenomena* ed E Brezin and J Zinn-Justin (Amsterdam: Elsevier) p 563
[14] Affleck I 1989 *Banff Lectures* (unpublished)
[15] Schlottmann P 1987 *Phys. Rev.* B **36** 5177
[16] Lieb E H and Wu F Y 1968 *Phys. Rev. Lett.* **20** 1445
[17] Belavin A A, Polyakov A A and Zamolodchikov A B 1984 *Nucl. Phys.* B **241** 333
See for a collection of reprints: Itzykson C, Saleur H and Zuber J-B (ed.) 1988 *Conformal Invariance and Applications in Statistical Mechanics* (Singapore: World Scientific)
[18] Cardy J L 1986 *Nucl. Phys.* B **270** [FS16] 186
[19] de Vega H J and Woynarovich F 1985 *Nucl. Phys.* B **251** [FS13] 439
Woynarovich F and Eckle H-P 1987 *J. Phys. A: Math. Gen.* **20** L97
[20] Pokrovskii S V and Tsvelik A M 1987 *Sov. Phys.–JETP* **66** 1275
[21] Suzuki J 1988 *J. Phys. A: Math. Gen.* **21** L1175
[22] Izergin A G, Korepin V E and Reshetikhin N Yu 1989 *J. Phys. A: Math. Gen.* **22** 2615
[23] Frahm H and Yu N C 1990 *J. Phys. A: Math. Gen.* **23** 2115
[24] Zhang F C and Rice T M 1988 *Phys. Rev.* B **37** 3759
[25] Sutherland B 1975 *Phys. Rev.* B **12** 3795
[26] Bares P A and Blatter G 1990 *Phys. Rev. Lett.* **64** 2567
[27] Kawakami N and Okiji A 1981 *Phys. Lett.* **86A** 483
[28] Woynarovich F 1989 *J. Phys. A: Math. Gen.* **22** 4243
[29] Woynarovich F, Eckle H-P and Truong T T 1989 *J. Phys. A: Math. Gen.* **22** 4027
[30] Blöte H W J, Cardy J L and Nightingale M P 1986 *Phys. Rev. Lett.* **56** 742
Affleck I 1986 *Phys. Rev. Lett.* **56** 746
[31] Imada M 1990 *J. Phys. Soc. Japan* **59** 4121
[32] Bogoliubov N M, Izergin A G and Korepin V E 1986 *Nucl. Phys.* B **275** 687
[33] Kawakami N and Okiji A 1990 *Prog. Theor. Phys.* Suppl. **101** 429
[34] Griffiths R B 1964 *Phys. Rev.* **133** A768
[35] Ogata M, Sugiyama T and Shiba H 1991 *Phys. Rev.* B **43** 8401
[36] Kadanoff L P and Brown A C 1979 *Ann. Phys., NY* **121** 318
Yang S-K 1987 *Nucl. Phys.* B **285** [FS19] 183
[37] Luttinger J M 1963 *J. Math. Phys.* **4** 1154
[38] Gutfreund H and Schick M 1968 *Phys. Rev.* **168** 418
[39] Luther A and Peschel I 1974 *Phys. Rev.* B **9** 2911
[40] Parola A and Sorella S 1990 *Phys. Rev. Lett.* **64** 1831
[41] Anderson P W 1990 *Phys. Rev. Lett.* **64** 1839
[42] Usuki T, Kawakami N and Okiji A 1989 *Phys. Lett.* **135A** 476
Kawakami N, Usuki T and Okiji A 1989 *Phys. Lett.* **137A** 287
[43] Shastry B S and Sutherland B 1990 *Phys. Rev. Lett.* **65** 243
[44] Kawakami N and Yang S-K 1990 *Phys. Rev. Lett.* **65** 3063